Equivalent definition of Riemann integrability

Definition 0.1. *$f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable if $\exists L \in \mathbb{R}$ such that for any $\dot{\mathcal{P}}$ satisfying $||\dot{\mathcal{P}}|| < \delta$,*

$$|\sum_j f(t_j)\ \Delta x_j - L| < \epsilon$$

where $t_j \in [x_j, x_{j+1}]$ are the tags.

Theorem 0.2. *(See textbook) If $f \in R[a, b]$, then f is bounded.*

Alternative approach:

For bounded function $f : [a, b] \to \mathbb{R}$, for any Partition $\mathcal{P}$, define

$$U(f, \mathcal{P}) = \sum_j M_j\ \Delta x_j, \quad L(f, \mathcal{P}) = \sum_j m_j\ \Delta x_j$$

where $M_j = \sup\{f(x) : x \in [x_j, x_{j+1}]\}$, $m_j = \inf\{f(x) : x \in [x_j, x_{j+1}]\}$.

Lemma 0.3. *We have for any partition $\mathcal{P}_1$,$\mathcal{P}_2$, we have*

$$L(f, \mathcal{P}_1) \le U(f, \mathcal{P}_2).$$

In particular, we have for any partition $\mathcal{P}$,

$$L(f, \mathcal{P}) \le \sup_{\mathcal{P}} L(f, \mathcal{P}) = \underline{\int_a^b} f \le \overline{\int_a^b} f = \inf_{\mathcal{P}} U(f, \mathcal{P}) \le U(f, \mathcal{P}).$$

Proof. By considering the new partition $\mathcal{P}_1 \cup \mathcal{P}_2$ and apply the monotonicity of $U(f, \mathcal{P})$ and $L(f, \mathcal{P})$. □

Definition 0.4. *For bounded function $f : [a, b] \to \mathbb{R}$, f is said to be integrable if there exists unique $A \in \mathbb{R}$ such that*

$$L(f, \mathcal{P}) \le A \le U(f, \mathcal{P}).$$

Or equivalently,

$$\underline{\int_a^b} f = \overline{\int_a^b} f.$$

Lemma 0.5. *For bounded function $f : [a, b] \to \mathbb{R}$, f is integrable if and only if $\forall \epsilon > 0$, $\exists \mathcal{P}$ such that*

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon.$$

Proof. It is clear from the definition of sup, inf and lemma 0.3 . □

The above two definitions are equivalent. By cauchy criterion in textbook, it suffices to show the followings. (In particular, $L = \int_a^b f = \underline{\int_a^b} f = \overline{\int_a^b} f$.)

Theorem 0.6. *For bounded function $f : [a, b] \to \mathbb{R}$, f is integrable if and only if* $\forall \epsilon > 0$, $\exists \delta > 0$ *such that*

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon$$

whenever $||\mathcal{P}|| < \delta$.

Proof. The "if" part is trivial.
Let $M > 0$ such that $|f| \leq M$. Let $\epsilon > 0$ be given, suffices to find the corresponding $\delta > 0$. By assumption, $\exists \tilde{\mathcal{P}}$ such that

$$U(f, \tilde{\mathcal{P}}) - L(f, \tilde{\mathcal{P}}) < \epsilon/2.$$

Say $\tilde{\mathcal{P}} : a = x_0 < x_1 < ... < x_N = b$. Choose $\delta > 0$ such that $\delta < \min\{\Delta x_1, \Delta x_2, ..\Delta x_N, \epsilon/8MN\}$. For any partition $\mathcal{P}$ such that $||\mathcal{P}|| < \delta$, $\mathcal{P} : y_0 = a < y_1 < ... < y_n = b$,

$$\begin{aligned}\sum_i (M_i - m_i)\Delta y_i &= \sum_{[y_i, y_{i+1}] \subset [x_j, x_{j+1}] \text{for some j}} (M_i - m_i)\Delta y_i \\ &+ \sum_{x_j \in [y_i, y_{i+1}] \text{for some j}} (M_i - m_i)\Delta y_i \\ &\leq U(f, \tilde{\mathcal{P}}) - L(f, \tilde{\mathcal{P}}) + 2MN \cdot ||\mathcal{P}|| \\ &< \epsilon/2 + 2MN \cdot \frac{\epsilon}{8MN} < \epsilon.\end{aligned}$$

□

Example 0.7. *Let $g : [0, 1] \to \mathbb{R}$ be a continuous function. Let $f : [0, 1] \to \mathbb{R}$ such that*

$$\begin{cases} f(x) & = 1 \ \text{ if } \ x = y_n = 1/n, \\ f(x) & = g(x) \text{ otherwise.} \end{cases}$$

Then f in integrable.

Proof. Since g is continuous on closed and bounded interval. g is bounded, and so is f. Let $M > 0$ such that

$$|f(x)| < M.$$

Let $\epsilon > 0$ be given. There exists $N = N(\epsilon) \in \mathbb{N}$ such that $1/n < \epsilon/8M$ for all $n > N$. Choose $x_0 = 0$, $x_1 = \epsilon/8M$. Thus,

$$\frac{1}{n} \in [x_0, x_1], \ \ \forall n > N.$$

Noted that there are only finitely many y_n outside $[x_0, x_1]$. By continuity of g, there exists $\delta' > 0$ such that whenever $|x - y| < \delta'$, $|g(x) - g(y)| < \epsilon/2$. Choose $\delta > 0$ such that $\delta < \delta'$ and $\delta < \epsilon/8MN$. Choose a partition on $[x_1, 1]$ such that $||\mathcal{P}|| < \delta$. Glue $\mathcal{P}$ and $[x_0, x_1]$ to form a partition $\mathcal{Q}$ on $[0,1]$. We have

$$\begin{aligned}
U(f,\mathcal{Q}) - L(f,\mathcal{Q}) &= \sum_j (M_j - m_j)\Delta x_j \\
&\leq \left(\sum_{[x_j,x_{j+1}]\ni y_k} + \sum_{y_k \notin [x_j,x_{j+1}]} \right) (M_j - m_j)\Delta x_j \\
&\leq 2M \cdot (x_1 - x_0) + 2MN\delta + \sum_j \Delta x_j \cdot \epsilon/2 \\
&\leq \epsilon \cdot (\frac{2M}{8M} + \frac{2MN}{8MN} + \frac{1}{2}) = \epsilon.
\end{aligned}$$

□